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INVESTIGATION OF PROCEDURES FOR AUTOMATIC RESONANCE EXTRACTION FROM NOISY TRANSIENT ELECTROMAGNETICS DATA

Final Report for Contract # N00014-80-C-0299

Volume II - Appendices

to

Office of Naval Research
800 North Quincy Street
Arlington, VA 22217

Attention: Dr. Henry Mullaney
Code 427

17 August 1981

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This volume consists of a series of appendices whose contents could not be included in the body of Volume I without breaking the continuity. The appendices describe topics that were studied during the course of the current contract but are only tangentially related to the theme of Volume I. Several references were made in Volume I to appendices in this volume for the details of specific methods.

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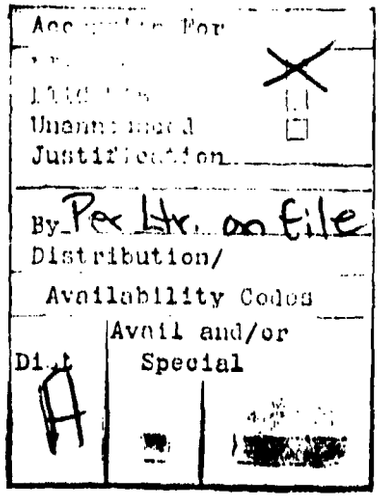
Volume II - Appendices

by

Jon R. Auton
Michael L. Van Blaricum



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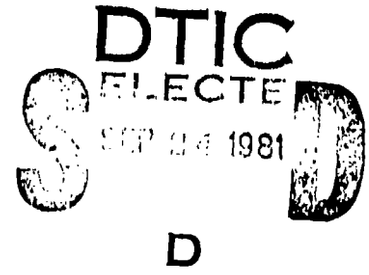

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INTRODUCTION TO VOLUME II

This volume consists of a series of appendices whose contents could not be included in the body of the report without breaking the continuity. The appendices describe topics that were studied during the course of the current contract but are only tangentially related to the theme of Volume I. Several references were made in Volume I to appendices in this volume for the details of specific methods.

The notation introduced in each appendix applies only to that appendix unless otherwise noted.

APPENDIX A

THE METHOD OF ITERATIVE GENERALIZED LEAST SQUARE

In Prony's method we have the difference equation

$$\sum_{p=0}^{N} \alpha_p \, I_{p+K} = 0 \quad , \quad K = 0,1,\ldots,\gamma-1 \qquad (A.1)$$

$$\gamma = M-N$$

This is usually solved as an inhomogeneous equation

$$\sum_{p=0}^{N-1} \alpha_p \, I_{p+K} = -I_{N+K} \quad , \quad K = 0,1,\ldots,\gamma-1 \qquad (A.2)$$

where α_N was set to one.

However we do not know the I_K's exactly. The measurements of the I_K's are Y_K as

$$Y_K = I_K + e_K \qquad (A.3)$$

where e_K is the error in the K^{th} sample.

Hence,

$$\sum_{p=0}^{N} \alpha_p \, Y_{p+K} = \sum_{p=0}^{N} \alpha_p \, e_{p+K} \quad . \qquad (A.4)$$

Rewriting the above gives

$$\sum_{p=0}^{N-1} \alpha_p \, Y_{p+K} = -Y_{N+K} + \sum_{p=0}^{N} \alpha_p \, e_{p+K} \quad . \qquad (A.5)$$

Letting $W_K = \sum_{p=0}^{N} \alpha_p \, e_{p+K}$ (A.6)

then

$$\sum_{p=0}^{N-1} \alpha_p \, Y_{p+K} = -Y_{N+K} + W_K \quad . \tag{A.7}$$

The W_K are the residuals but from (A.6) the residuals are correlated and a least squares solution to (A.6) will give biased estimates.

When we use least squares to minimize the residuals by

$$\frac{\partial}{\partial \alpha_m} \sum_{K=0}^{\gamma-1} W_K^2 = 0 \quad , \quad m=0,1,\ldots,N-1 \tag{A.8}$$

we obtain the final expression

$$\sum_{p=0}^{N-1} \alpha_p \sum_{K=0}^{\gamma-1} Y_{p+K} \, Y_{m+K} = - \sum_{K=0}^{\gamma-1} Y_{N+K} \, Y_{m+K} \quad , \quad m=0,\ldots,N-1 \quad . \tag{A.9}$$

We will get biased estimates for the α_p.

One way to correct this is to use a method known as iterative generalized least squares. Rewrite (A.7) as

$$\sum_{p=0}^{N} \alpha_p \, Y_{p+K} = W_K \quad , \quad K=0,1,\ldots,\gamma-1 \quad . \tag{A.8}$$

Now define new notation for the sake of convenience. First introduce the shift operator q defined so that

$$\begin{aligned} q f_K &= f_{K+1} \\ q^2 f_K &= f_{K+2} \end{aligned} \tag{A.9}$$

The polynomial operator A(q) is defined as

$$A(q) = \sum_{m=0}^{N} a_m \, q^m \quad . \tag{A.10}$$

Now (A.8) can be written as

$$A(q)\ Y_K = W_K \quad , \quad \text{let } \alpha = a \quad . \tag{A.11}$$

In matrix form (A.8) is

$$\begin{bmatrix} Y_0 & Y_1 & Y_2 \\ Y_1 & Y_2 & Y_3 \\ Y_2 & Y_3 & Y_4 \\ Y_3 & Y_4 & Y_5 \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} W_0 \\ W_1 \\ W_2 \\ \end{bmatrix}$$

and (A.11) is

$$\begin{bmatrix} q^0Y_0 & q^1Y_0 & q^2Y_0 \\ q^0Y_1 & q^1Y_1 & q^2Y_1 \\ q^0Y_2 & q^1Y_2 & q^2Y_2 \\ q^0Y_3 & q^1Y_3 & q^2Y_3 \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} W_0 \\ W_1 \\ W_2 \\ W_3 \end{bmatrix} .$$

Now let us operate on W_K with the polynomial filter

$$B(q)\ W_K = n_K \tag{A.12}$$

to give n_K which is an uncorrelated noise sequence (random variables) so that

$$B(q)\ A(q)\ Y_K = n_K \quad .$$

Let A commute with B so that

$$A(q)\ B(q)\ Y_K = n_K \tag{A.13}$$

and define

$$\tilde{Y}_K = B(q)\ Y_K \tag{A.14}$$

so that

$$A(q)\ \tilde{Y}_K = n_K \quad . \tag{A.15}$$

Now equation (A.15) has uncorrelated residuals and hence the solution should converge to unbiased estimates.

The iterative procedure is therefore as follows:

1. Solve equation (A.11) using the normal least squares procedure. This gives an estimate of A(q) which can be thought of as the α's of (A.8). Call that estimate $\hat{A}(q)$.
2. Substitute $\hat{A}(q)$ into (A.11) to produce an estimate of the residuals $\hat{W}_K$.
3. Use $\hat{W}_K$ in (A.12) to make a least squares estimate of $\hat{B}(q)$.
4. Calculate $\hat{\tilde{Y}}_K$ in (A.14).
5. Use the $\hat{\tilde{Y}}_K$ to make a new least squares estimate $\hat{\hat{A}}_K(q)$.
6. Continue this procedure until

   $\sum_{p=0}^{\gamma-1} W_K^{\,2}$ no longer decreases with the next iteration.

The question now is:

How does one obtain $\hat{B}(q)$ of step 3 above?

Assume we have used the least squares process once to find the coefficients α_p.

Next calculate the γ residuals W_K as

$$\sum_{p=0}^{N} \alpha_p \, Y_{p+K} = W_K \quad , \quad K=0,1,\ldots,\gamma-1 \quad .$$

The results of steps 1 and 2 above gives a γ dimensional vector of the residuals W_K.

Calculate the autovariance function of the W_K by

$$r(u) = \frac{c(u)}{c(o)} \quad , \quad u=0,1,\ldots,\gamma-1$$

where

$$c(u) = \frac{1}{\gamma} \sum_{i=1}^{\gamma-u} (W_i - \bar{W}) \, (W_{i+u} - \bar{W}) \quad , \quad u=0,1,\ldots,\gamma-1$$

and

$$\bar{W} = \frac{1}{\gamma} \sum_{i=1}^{\gamma} W_i \quad .$$

Hence the autocovariance function r(u) is defined

$$r(u) = \frac{\sum_{i=1}^{\gamma-u} (W_i - \bar{W}) \, (W_{i+u} - \bar{W})}{\sum_{i=1}^{\gamma} (W_i - \bar{W})^2} \quad , \quad y=0,1,\ldots,\gamma-1 \quad .$$

We now have a vector R, γ long.

We want to test this R vector for whiteness of the residuals W_K. The standard deviation of a single autocovariance function estimate is

$$\frac{1}{\sqrt{\gamma}} .$$

The 95% confidence limits for a single autocovariance sample are

$$r(u) \pm \frac{1}{\sqrt{\gamma}} \; 1.96 \quad .$$

Now supposedly if 95% of the samples r(u) are less than $\frac{1.96}{\sqrt{\gamma}}$ then we have 95% confidence that the noise is white.

If the r(u) flunks the whiteness test which will happen the first few times we iterate we must then go back and whiten the residuals W_K.

From Equation (A.12) we want

$$B(q)\, W_K = n_K \quad .$$

Remembering that $B(q) = \sum_{m=0}^{L} b_m \, q^m$,

$$q f_K = f_{K+1}$$

and the n_K are white random deviates.

Thus we have a matrix equation of the form Wb = n.

As an example, assume L = 2 and K = 3. Then we get

$$\begin{bmatrix} q^0 W_0 & q^1 W_0 & q^2 W_0 \\ q^0 W_1 & q^1 W_1 & q^2 W_1 \\ q^0 W_2 & q^1 W_2 & q^2 W_2 \\ q^0 W_3 & q^1 W_3 & q^2 W_3 \end{bmatrix} \quad \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} n_0 \\ n_1 \\ n_2 \\ n_3 \end{bmatrix}$$

or

$$\begin{bmatrix} W_0 & W_1 & W_2 \\ W_1 & W_2 & W_3 \\ W_2 & W_3 & W_4 \\ W_3 & W_4 & W_5 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} n_0 \\ n_1 \\ n_2 \\ n_3 \end{bmatrix}$$

Solving $Wb = n$ for the b_n by using least squares gives

$$W^T W b = W^T n$$

or

$$b = (W^T W)^{-1} W^T n \quad .$$

Therefore the new $\tilde{Y}_K$ previously defined as

$$\tilde{Y}_K = B(q)\, Y_K$$

can be written as

$$\tilde{Y}_K = b_0 Y_K + b_1 Y_{K+1} + b_2 Y_{K+2} + \ldots + B_L Y_{K+L} \quad .$$

APPENDIX B

EFFECT OF INCREASING MODEL ORDER IN PRONY'S METHOD

A study of the effect of increasing model order in Prony's method has been made and the results are presented here along with some preliminary conclusions. In past investigations of Prony's method it has been noted that increasing the model order above the known order of the waveform improves Prony's method's ability to estimate accurately the true poles. Although the accuracy of the pole estimates is improved, a side effect of this procedure is the problem of distinguishing between true poles and those poles that simply fit to the noise and have no relation to the information that is to be extracted from the waveform.

In this study we relate the inaccuracy or bias of the parameter estimates to the amplitude of the residuals of the least-square Prony procedure. We assume that the inhomogenous solution (defined in Volume I, Section 2) is used to find a parameter vector. The term "residuals" is identical to "equation error". In this appendix, N is the number of poles modeled and M is the number of samples. When the residuals are zero the least-squares Prony's method either has reduced to curve-fitting Prony's method (M=2N) or is processing a noise-free waveform at the proper model order. In this case, the pole estimates are quite accurate and are free of bias. Conversely the bias in Prony's method is directly related to the magnitude of the residuals.

Tables B.1 through B.9 display the effect of increasing model order at three different noise levels. σ_N is the standard deviation of the noise. $\bar{\sigma}_R$ is the average standard deviation of the residuals over ten Monte Carlo runs. Each table shows the true parameter values in the first column, the average parameter values over ten Mone Carlo runs, and the variance of each parameter in the third column. For all nine cases the time window size is kept constant. Thus, M = 100 and ΔT = 0.13 seconds.

The three pole pairs shown under the average column were obtained by searching all N poles to find the pairs that were closest to the true poles in value. In general, of the extra poles not reported here, N-6, have residues two orders of magnitude below those of the true poles.

From the results of Tables B.1 through B.9, the following observations can be made:

1. Prony's method seems to guarantee good results if $\sigma_R < \sigma_N$.
2. $\sigma_R < \sigma_N$ seems to occur when $4N > M$.

That good results begin at $\sigma_R = \sigma_N$ is not surprising. In this case one would expect that the residuals are beginning to approximate the noise and that the "modeled" waveform approaches the uncorrupted waveform. But the observation that $\sigma_R = \sigma_N$ approximately when $4N = M$ has no obvious explanation. This observation would obviously not hold true if we were to apply Prony's method to a noise-free waveform. In this case, the residuals would drop to zero as soon as the model order reached or exceeded the order of the waveform. We might surmise then that this occurrance at $4N = M$ is attributable to the nature of the uncorrelated noise.

There is no simple explanation for improved accuracy at higher orders. At least two factors seem to be involved:

1. Increasing the length of the parameter vector (by increasing the order) is an effective treatment for the dense sampling problem (which is described in Volume I, Section 3 of this report.)
2. Making the data matrix more nearly square must reduce the magnitude of the residuals. Smaller residuals implies smaller bias in the parameters.

Table B.1. Model order study M=100, N=12, σ_N=.100, $\bar{\sigma}_R$=.1808.

TRUE	AVERAGE	σ^2	
-8.200000E-02	-6.812908E-01	7.546325E-03	REAL PARTS OF POLES
-8.200000E-02	-6.812908E-01	7.546325E-03	
-1.470000E-01	-6.642028E-01	5.821274E-03	
-1.470000E-01	-6.642028E-01	5.821274E-03	
-1.880000E-01	-3.329216E-01	2.371370E-03	
-1.880000E-01	-3.329216E-01	2.371370E-03	
9.260000E-01	0.	0.	IMAG. PARTS OF POLES
-9.260000E-01	0.	0.	
2.874000E+00	2.341076E+00	3.239736E-03	
-2.874000E+00	-2.341076E+00	3.239736E-03	
4.835000E+00	4.844311E+00	6.047468E-04	
-4.835000E+00	-4.844311E+00	6.047468E-04	
1.000000E+00	1.109395E+00	7.509193E-02	MAGNITUDES OF RESIDUES
1.000000E+00	1.109395E+00	7.509193E-02	
1.000000E+00	1.144378E+00	1.351987E-02	
1.000000E+00	1.144378E+00	1.351987E-02	
1.000000E+00	1.189903E+00	6.474729E-03	
1.000000E+00	1.189903E+00	6.474729E-03	
0.	4.419111E-15	4.077692E-28	RADIAN PHASE OF RESIDUES
0.	4.419111E-15	4.077692E-28	
0.	1.507967E-01	1.508953E-02	
0.	-1.507967E-01	1.508953E-02	
0.	-1.701596E-01	5.476887E-03	
0.	1.701596E-01	5.476887E-03	

Table B.2. Model order study M=100, N=20, σ_N=.100, $\bar{\sigma}_R$=.124.

TRUE	AVERAGE	σ^2	
-8.200000E-02	-9.069550E-02	6.905491E-05	REAL PARTS OF POLES
-8.200000E-02	-9.069550E-02	6.905491E-05	
-1.470000E-01	-1.571783E-01	1.083155E-04	
-1.470000E-01	-1.571783E-01	1.083155E-04	
-1.880000E-01	-1.951637E-01	1.278673E-04	
-1.880000E-01	-1.951637E-01	1.278673E-04	
9.260000E-01	9.289997E-01	3.657123E-05	IMAG. PARTS OF POLES
-9.260000E-01	-9.289997E-01	3.657123E-05	
2.874000E+00	2.883251E+00	1.011044E-04	
-2.874000E+00	-2.883251E+00	1.011044E-04	
4.835000E+00	4.840251E+00	1.176039E-04	
-4.835000E+00	-4.840251E+00	1.176039E-04	
1.000000E+00	1.027859E+00	2.037443E-03	MAGNITUDES OF RESIDUES
1.000000E+00	1.027859E+00	2.037443E-03	
1.000000E+00	1.036457E+00	2.199189E-03	
1.000000E+00	1.036457E+00	2.199189E-03	
1.000000E+00	1.018026E+00	1.059788E-03	
1.000000E+00	1.018026E+00	1.059788E-03	
0.	-1.230493E-02	9.620293E-04	RADIAN PHASE OF RESIDUES
0.	1.230493E-02	9.620293E-04	
0.	-1.928778E-02	2.042011E-03	
0.	1.928778E-02	2.042011E-03	
0.	-5.326322E-03	2.024303E-03	
0.	5.326322E-03	2.024303E-03	

Table B.3. Model order study M=100, N=36, σ_N=.100, $\bar{\sigma}_R$=.0678.

TRUE	AVERAGE	σ^2	
-8.200000E-02	-8.169175E-02	3.056657E-05	REAL PARTS OF POLES
-8.200000E-02	-8.169175E-02	3.056657E-05	
-1.470000E-01	-1.483538E-01	1.172273E-04	
-1.470000E-01	-1.483538E-01	1.172273E-04	
-1.880000E-01	-1.842590E-01	1.861397E-04	
-1.880000E-01	-1.842590E-01	1.861397E-04	
9.260000E-01	9.266480E-01	2.394936E-05	IMAG. PARTS OF POLES
-9.260000E-01	-9.266480E-01	2.394936E-05	
2.874000E+00	2.874836E+00	5.301869E-05	
-2.874000E+00	-2.874836E+00	5.301869E-05	
4.835000E+00	4.836128E+00	4.450036E-05	
-4.835000E+00	-4.836128E+00	4.450036E-05	
1.000000E+00	9.931078E-01	1.325078E-03	MAGNITUDES OF RESIDUES
1.000000E+00	9.931078E-01	1.325078E-03	
1.000000E+00	1.013720E+00	4.119833E-03	
1.000000E+00	1.013720E+00	4.119833E-03	
1.000000E+00	9.891063E-01	6.276759E-03	
1.000000E+00	9.891063E-01	6.276759E-03	
0.	-3.563147E-03	8.553559E-04	RADIAN PHASE OF RESIDUES
0.	3.563147E-03	8.553559E-04	
0.	1.445358E-03	1.790225E-03	
0.	-1.445358E-03	1.790225E-03	
0.	-1.402673E-02	7.134540E-04	
0.	1.402673E-02	7.134540E-04	

Table B.4. Modal order study M=100, N=8, σ_N=.00100, $\bar{\sigma}_R$=.007098.

TRUE	AVERAGE	σ^2	
-8.200000E-02	-4.746023E-01	6.871508E-03	REAL PARTS OF POLES
-8.200000E-02	-4.746023E-01	6.871508E-03	
-1.470000E-01	-3.237075E-01	1.154383E-03	
-1.470000E-01	-3.237075E-01	1.154383E-03	
-1.880000E-01	-1.994993E-01	2.614657E-03	
-1.880000E-01	-1.994993E-01	2.614657E-03	
9.260000E-01	8.950305E-01	8.704512E-04	IMAG. PARTS OF POLES
-9.260000E-01	-8.950305E-01	8.704512E-04	
2.874000E+00	2.874394E+00	2.601685E-04	
-2.874000E+00	-2.874394E+00	2.601685E-04	
4.835000E+00	4.843241E+00	3.869888E-05	
-4.835000E+00	-4.843241E+00	3.869888E-05	
1.000000E+00	1.737185E+00	9.114386E-03	MAGNITUDES OF RESIDUES
1.000000E+00	1.737185E+00	9.114386E-03	
1.000000E+00	1.223820E+00	1.481720E-03	
1.000000E+00	1.223820E+00	1.481720E-03	
1.000000E+00	9.028370E-01	4.550050E-04	
1.000000E+00	9.028370E-01	4.550050E-04	
0.	2.022803E-01	7.235554E-03	RADIAN PHASE OF RESIDUES
0.	-2.022803E-01	7.235554E-03	
0.	2.298335E-01	3.769424E-03	
0.	-2.298335E-01	3.769424E-03	
0.	1.280894E-01	2.455785E-03	
0.	-1.280894E-01	2.455785E-03	

Table B.5. Model order study M=100, N=12, σ_N=.00100, $\bar{\sigma}_R$=.00300.

TRUE	AVERAGE	σ^2	
-8.200000E-02	-8.334790E-02	3.819838E-05	REAL PARTS OF POLES
-8.200000E-02	-8.334790E-02	3.819838E-05	
-1.470000E-01	-1.482045E-01	3.307556E-05	
-1.470000E-01	-1.482045E-01	3.307556E-05	
-1.880000E-01	-1.879587E-01	1.855490E-07	
-1.880000E-01	-1.879587E-01	1.855490E-07	
9.260000E-01	9.263227E-01	3.054208E-05	IMAG. PARTS OF POLES
-9.260000E-01	-9.263227E-01	3.054208E-05	
2.874000E+00	2.874264E+00	2.313338E-05	
-2.874000E+00	-2.874264E+00	2.313338E-05	
4.835000E+00	4.835043E+00	8.757051E-07	
-4.835000E+00	-4.835043E+00	8.757051E-07	
1.000000E+00	1.005441E+00	9.224108E-05	MAGNITUDES OF RESIDUES
1.000000E+00	1.005441E+00	9.224108E-05	
1.000000E+00	1.003853E+00	4.718098E-05	
1.000000E+00	1.003853E+00	4.718098E-05	
1.000000E+00	9.994317E-01	4.084030E-05	
1.000000E+00	9.994317E-01	4.084030E-05	
0.	-1.110413E-03	8.015323E-05	RADIAN PHASE OF RESIDUES
0.	1.110413E-03	8.015323E-05	
0.	8.180271E-04	4.066317E-05	
0.	-8.180270E-04	4.066317E-05	
0.	1.859155E-03	9.113711E-05	
0.	-1.859155E-03	9.113711E-05	

Table B.6. Model order study M=100, N=20, σ_N=.001, σ_R=.001246.

TRUE	AVERAGE	σ^2	
-8.200000E-02	-8.199855E-02	6.959435E-09	REAL PARTS OF POLES
-8.200000E-02	-8.199855E-02	6.959435E-09	
-1.470000E-01	-1.470155E-01	1.034647E-08	
-1.470000E-01	-1.470155E-01	1.034647E-08	
-1.880000E-01	-1.880358E-01	1.121223E-08	
-1.880000E-01	-1.880358E-01	1.121223E-08	
9.260000E-01	9.260079E-01	3.504264E-09	IMAG. PARTS OF POLES
-9.260000E-01	-9.260079E-01	3.504264E-09	
2.874000E+00	2.874023E+00	9.536052E-09	
-2.874000E+00	-2.874025E+00	9.536052E-09	
4.835000E+00	4.834994E+00	1.374833E-08	
-4.835000E+00	-4.834994E+00	1.374833E-08	
1.000000E+00	9.999503E-01	2.130885E-07	MAGNITUDES OF RESIDUES
1.000000E+00	9.999503E-01	2.130885E-07	
1.000000E+00	1.000110E+00	2.091616E-07	
1.000000E+00	1.000110E+00	2.091616E-07	
1.000000E+00	1.000065E+00	9.094737E-08	
1.000000E+00	1.000065E+00	9.094737E-08	
0.	-4.401502E-05	9.035431E-08	RADIAN PHASE OF RESIDUES
0.	4.401502E-05	9.035431E-08	
0.	-6.973260E-05	2.004572E-07	
0.	6.973260E-05	2.004572E-07	
0.	-2.903913E-05	2.259596E-07	
0.	2.903913E-05	2.259596E-07	

Table B.7. Model order study M=100, N=8, σ_N=.0100, $\bar{\sigma}_R$=.0394.

TRUE	AVERAGE	σ^2	
-8.200000E-02	-4.816941E-01	4.313874E-03	REAL PARTS OF POLES
-8.200000E-02	-4.816941E-01	4.313874E-03	
-1.470000E-01	-4.931013E-01	1.633214E-03	
-1.470000E-01	-4.931013E-01	1.633214E-03	
-1.880000E-01	-2.603545E-01	4.518706E-05	
-1.880000E-01	-2.603545E-01	4.518706E-05	
9.260000E-01	0.	0.	IMAG. PARTS OF POLES
-9.260000E-01	0.	0.	
2.874000E+00	2.444875E+00	6.252500E-04	
-2.874000E+00	-2.444875E+00	6.252500E-04	
4.835000E+00	4.820957E+00	1.699293E-04	
-4.835000E+00	-4.820957E+00	1.699293E-04	
1.000000E+00	8.254603E-01	2.930682E-02	MAGNITUDES OF RESIDUES
1.000000E+00	8.254603E-01	2.930682E-02	
1.000000E+00	1.154058E+00	3.568307E-03	
1.000000E+00	1.154058E+00	3.568307E-03	
1.000000E+00	1.112501E+00	4.643365E-05	
1.000000E+00	1.112501E+00	4.643365E-05	
0.	2.991933E-15	6.596341E-25	RADIAN PHASE OF RESIDUES
0.	2.991933E-15	6.596341E-28	
0.	1.625033E-01	2.729688E-03	
0.	-1.625033E-01	2.729688E-03	
0.	-1.075143E-01	9.082337E-04	
0.	1.075143E-01	9.082337E-04	

Table B.8. Model order study M=100, N=12, σ_N=.0100, $\bar{\sigma}_R$=.0295.

TRUE	AVERAGE	σ^2	
-8.200000E-02	-2.353161E-01	2.859268E-03	REAL PARTS OF POLES
-8.200000E-02	-2.353161E-01	2.859268E-03	
-1.470000E-01	-2.407465E-01	5.518517E-04	
-1.470000E-01	-2.407465E-01	5.518517E-04	
-1.880000E-01	-1.934551E-01	1.094043E-05	
-1.880000E-01	-1.934561E-01	1.094043E-05	
9.260000E-01	9.408873E-01	3.593158E-04	IMAG. PARTS OF POLES
-9.260000E-01	-9.408873E-01	3.593158E-04	
2.874000E+00	2.899693E+00	1.687968E-04	
-2.874000E+00	-2.899693E+00	1.687968E-04	
4.835000E+00	4.842353E+00	1.122618E-04	
-4.835000E+00	-4.842353E+00	1.122618E-04	
1.000000E+00	1.426050E+00	1.708038E-02	MAGNITUDE OF RESIDUES
1.000000E+00	1.426050E+00	1.708038E-02	
1.000000E+00	1.211073E+00	2.967772E-03	
1.000000E+00	1.211073E+00	2.967772E-03	
1.000000E+00	9.755313E-01	4.150653E-04	
1.000000E+00	9.755313E-01	4.150653E-04	
0.	2.712849E-03	7.626122E-03	RADIAN PHASE OF RESIDUES
0.	-2.712849E-03	7.626122E-03	
0.	7.308093E-02	3.766332E-03	
0.	-7.308093E-02	3.766332E-03	
0.	1.025580E-01	8.128753E-04	
0.	-1.025580E-01	8.128753E-04	

Table B.9. Model order study M=100, N=20, σ_N=.0100, $\bar{\sigma}_R$=.0124.

TRUE	AVERAGE	σ^2	
-8.200000E-02	-8.206914E-02	6.936625E-07	REAL PARTS OF POLES
-8.200000E-02	-8.206914E-02	6.936625E-07	
-1.470000E-01	-1.472350E-01	1.036735E-05	
-1.470000E-01	-1.472350E-01	1.036735E-05	
-1.880000E-01	-1.883931E-01	1.133451E-06	
-1.880000E-01	-1.883931E-01	1.133451E-05	
9.260000E-01	9.260993E-01	3.514693E-07	IMAG. PARTS OF POLES
-9.260000E-01	-9.260993E-01	3.514693E-07	
2.874000E+00	2.874313E+00	9.613474E-07	
-2.874000E+00	-2.874313E+00	9.613474E-07	
4.835000E+00	4.834993E+00	1.354101E-05	
-4.835000E+00	-4.834993E+00	1.354101E-06	
1.000000E+00	9.998171E-01	2.123205E-05	MAGNITUDE OF RESIDUES
1.000000E+00	9.998171E-01	2.123205E-05	
1.000000E+00	1.001344E+00	2.099578E-05	
1.000000E+00	1.001344E+00	2.099578E-05	
1.000000E+00	1.000757E+00	9.211955E-05	
1.000000E+00	1.000757E+00	9.211955E-05	
0.	-5.161911E-04	9.070248E-05	RADIAN PHASE OF RESIDUES
0.	5.161911E-04	9.070248E-05	
0.	-8.186032E-04	2.000169E-05	
0.	8.186032E-04	2.000169E-05	
0.	-3.094731E-04	2.238498E-05	
0.	3.094731E-04	2.238498E-05	

APPENDIX C

THE ADAPTIVE METHOD FOR RESONANCE ESTIMATION

The adaptive method results from the generalized scheme of Volume I, Section 2, under the following assumptions:

1. $\alpha_o = 1$ and $\beta_o = 0$.
2. $F_i = \frac{z}{z-z_i}$, $i = 1,\ldots n$ and $F_o = 1$.
3. The model input is a unit sample at $k = 0$ (discrete impulse).

The unique feature of the method is that the filter poles, z_i, may be adjusted to any value in the Z-plane. An adaptive technique for adjusting the filters consists of first initializing the filters to arbitrary values in the Z-plane and repeating the following steps:

1. Find an estimate of the process transfer function using the current filters in the model.
2. Set each filter pole to one pole of the estimated transfer function.

This procedure is repeated until α_i approach zero. The poles of the process can be estimated during the course of iteration by

$$\hat{z}_i = \frac{z_i}{1 + \alpha_i}$$

removing the need for finding the roots of a polynomial. The filter poles are updated to $\hat{z}_i$ on each iteration. When the α_i approach zero the pole updating ceases and the method converges. At each iteration, the α_i are found using any of the techniques for finding a parameter vector described in Volume I, Section 2. Perhaps the simplest method is to choose the parameter vector as the weakest eigenvector of $\Omega^*\Omega$. At convergence the S-plane poles, s_i can be obtained from the filter poles by

$$s_i = \frac{\ln z_i}{T}$$

and the $A_i = \beta_i$.

The adaptive filtering method seems to be a very useful technique because it uses what seems to be optimal filters, band pass filters, for estimating pole locations. In addition it presents a measure of the error and iteratively adapts the filters until the error is at the level where it is desired.

As an example of the use of the adaptive filtering method consider the data shown in Figure C.1. These data were numerically generated by using the time domain computer code TWTD [C.1]. The structure modeled by TWTD was a thin cylindrical scatterer. The signal was contaminated with noise to give a 15 dB signal-to-noise ratio. Figure C.2 shows the resulting poles from five Monte Carlo trials.

The following statements can be made about the adaptive method after studying it.

The adaptive method is a new method which, in many cases, provides excellent pole estimates under difficult conditions. The method is unique in that a solution to a polynomial is not required to find estimates of the process poles. the method, in effect, "swallows" the polynomial solver in its own iterative pole-searching scheme.

Attempts to analyze waveforms consisting of highly damped exponential components, such as the transient responses of a sphere, have not been successful. The adaptive method does not converge for waveforms which display double pole characteristics, that is, waveforms with components of the form t exp(st). Slight modifications to the adaptive method might allow the analysis of such waveforms.

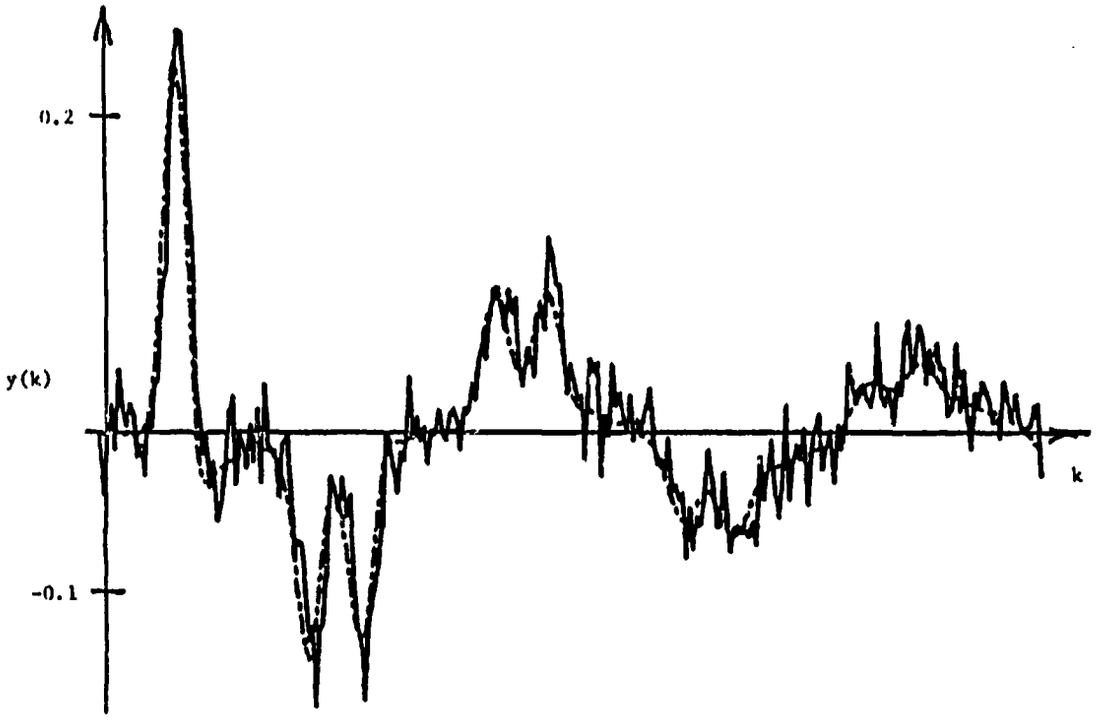

Figure C.1. Noise-contaminated offset-driven TWTD waveform, SNR = 15 dB. The uncontaminated waveform is plotted under the noisy waveform for comparison.

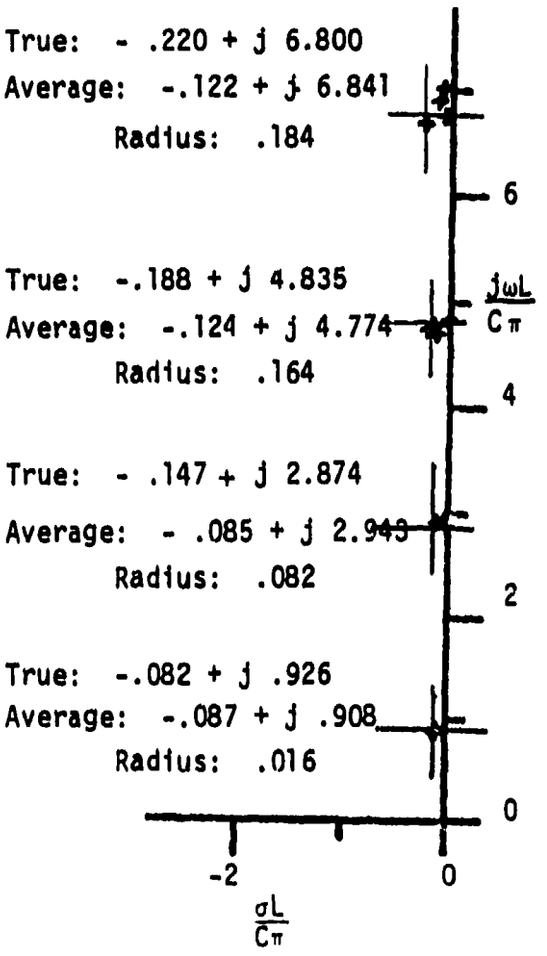

Figure C.2. Pole estimates obtained with the adaptive method.

References

[C.1] M.L. Van Blaricum, "A Numerical Technique for the Time-Dependent Solution of Thin-Wire Structure with Multiple Junctions", Electromagnetics Laboratory Report, 73-15, University of Illinois, Urbana, Illinois, December 1973.

APPENDIX D

THE PENCIL-OF-FUNCTIONS METHOD

The pencil-of-functions method results from the generalized model described in Volume I, Section 2 with the following assumptions:

1. $\alpha_0 = 1$ and $\beta_0 = 0$.
2. $F_i = F_i(s) = (1/s)^i$ (cascaded continuous integrators).

This method is not very easy to implement on a digital computer since the continuous-time integrators cannot be implemented exactly by any algorithm. When approximate integrators are cascaded, as they are in the pencil-of-functions method, large errors can be quickly accomulated and the intended result after a number of integrations destroyed.

This difficulty can be resolved by cascading discrete integrators to obtain filters with pulse transfer functions given by:

$$F_i = F_i(z) = \left(\frac{z}{z-1}\right)^i = \left(\frac{1}{Z}\right)^i$$

which can be implemented on a digital computer with no error by using difference equations. The variable Z is defined as

$$Z = 1 - \frac{1}{z}$$

and z is the z-transform variable.

When discrete integrators are used the discrete pencil-of-functions method results. The poles of the pulse transfer function of the process or waveform may be estimated as

$$z_i = \frac{1}{1-Z_i}$$

where Z_i is the i^{th} zero of

$$z^n + \alpha_1 z^{n-1} + \ldots + \alpha_n = Q \qquad (D.1)$$

In the original pencil-of-functions method the α_i were found by using Jain's method for constructing the parameter vector which is described in Volume I, Section 2. With Jain's method

$$\alpha_i = \sqrt{\frac{\Delta_{ii}}{\Delta_{oo}}} \ , \ i = 0, 1, \ldots, n;$$

where Δ_{ij}, in this case, is the element at the i^{th} row and j^{th} column of adj $\Omega^*\Omega$. Other methods can be used to estimate the parameter vector.

The S-plane estimates of the poles, s_i, are related to the zeros of (D.1) by

$$s_i = \frac{\ln(1-Z_i)}{T} \ .$$

One difficulty with the pencil-of-functions method is related to the attenuation of the higher frequency modes of the process output by the repeated integrations applied to the output waveform. It can be verified that an integrator is simply a first order filter whose Laplace transfer function has a pole at the origin in the S-plane. Such a filter tends to suppress the higher frequencies present at its input. The higher frequency suppression phenomenon is illustrated in Figure D.1. Normally, when an exponential function is integrated repeatedly, components of power of time exist in the higher integrals as well as the original exponential function components. In Figure D.4 the components of powers of time have been subtracted from the integrated waveforms in order to make the attenuation of the higher modes more evident. The first waveform is a hypothetical waveform provided for analysis. The waveforms that follow are the integrals of increasing order of the first waveform and display the increasing dominance of the fundamental mode or mode of lowest frequency. Further integrations yield nearly identical waveforms. The integrated waveforms tend to become linearly dependent

waveforms. The integrated waveforms tend to become linearly dependent at higher model orders. The matrix $\Omega^*\Omega$ then tends to singularity and the method becomes unstable. The suppression phenomenon occurs in both the discrete and continuous methods. In fact, even for very modest model orders, the method can become numerically ill-conditioned to a degree that special care must be taken to assure accurate inversion of $\Omega^*\Omega$.

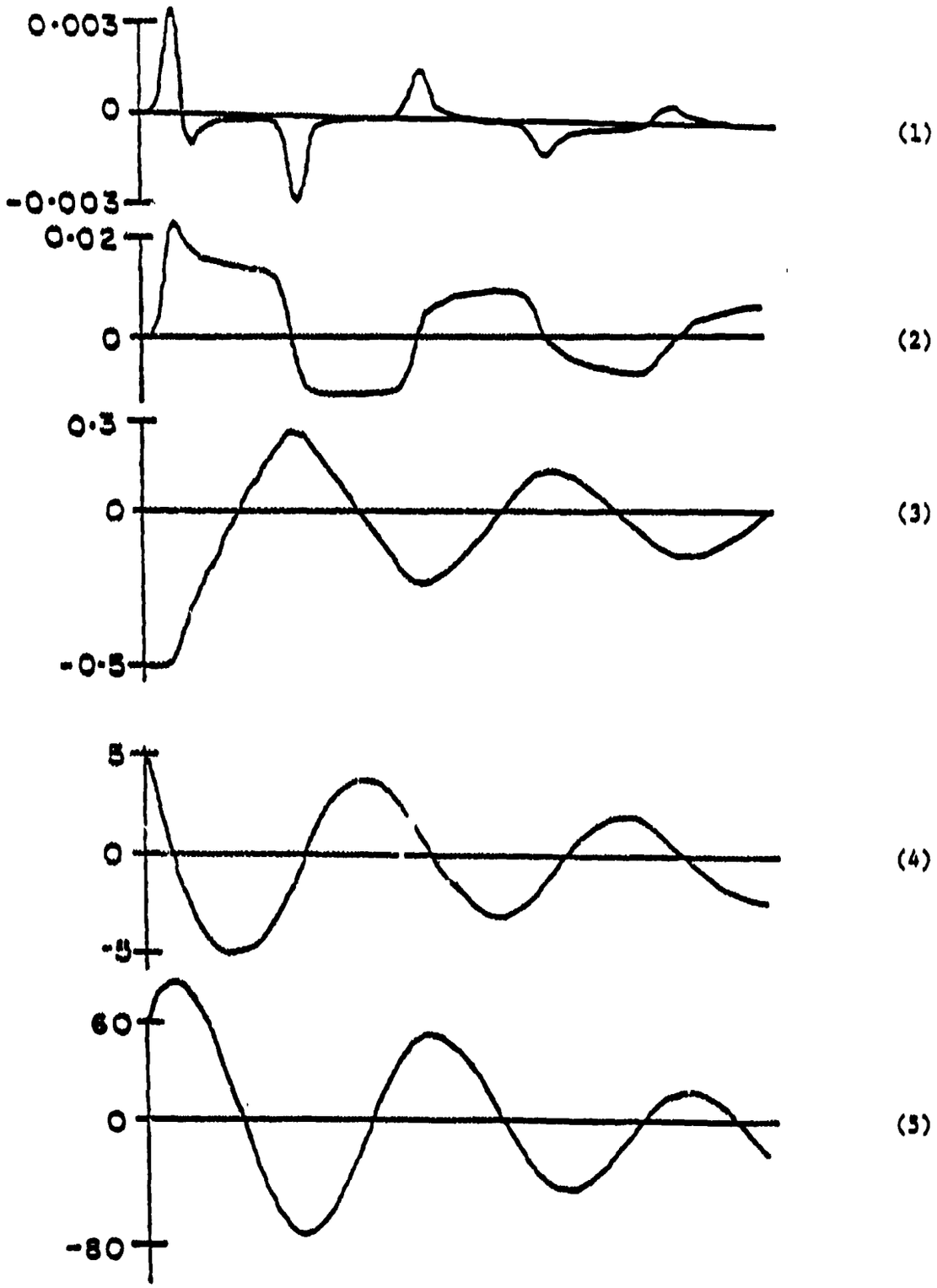

Figure D.4 Example of higher frequency suppression phenomenon seen in the pencil-of-functions method.

APPENDIX E

METHOD OF REDUNDANT AVERAGING

The redundant averaging scheme is a preprocessing scheme that attempts to combine redundant data (that is, more data than are necessary to determine the parameters) in a way that avoids the bias introduced by using a least-squares scheme to combine redundant data. The method attempts to transform a raw waveform of more than 2N samples where N is the model order into a preprocessed waveform of exactly 2N samples by averaging within the waveform. The averaging can be done so that the expectations of the poles of the preprocessed waveform are equivalent to the expectations of the poles of the raw waveform provided the additive noise on the raw waveform is zero mean and uncorrelated between sucessive samples. The preprocessed waveform may then be processed with curve-fitting Prony's method to avoid the bias of the least-squares procedure.

The most general description of the redundant averaging scheme can be stated simply as

$$X_k = \sum_{j=0}^{N_A-1} y_{j+kN_s}, \quad k=0,1,\ldots,2N-1$$

where N is the model order, X_k denotes the k^{th} sample of the preprocessed waveform, and y_k denotes the k^{th} sample of the raw waveform. In order to limit this description to the essential features of the redundant averaging scheme, N_A, the number of averages, and N_s, the decimation epoch, are not explicitly defined here. These parameters are choosen as desired but usually in a way that would produce a desirable preprocessing mode in some sense. For instance, the value for the decimation epoch might be chosen on the basis of the maximum frequency, ω_m, present in the raw waveform (if this information is available) and the number of averages might be chosen so that every sample in the raw waveform is used once the value of the decimation epoch is set. Hence the sampling interval, Δt, for the preprocessed waveform could be obtained as

$$\Delta t = N_s \, \Delta t_{raw}$$

where N_s = Integer Part $\left[\frac{\pi}{\omega_m \, \Delta t_{raw}}\right]$.

The number of averages, N_A, can be computed by

$$N_A = M - N_s \, (2N-1)$$

where M is the number of samples in the raw waveform and N is the order desired. It should be noted that the above method for determining N_A and N_s is not the only possible technique.

The redundant-averaging procedure is effectively two operations:

1. Apply a low-pass moving-average filter, A(z), to the raw waveform.
2. Decimate the filtered waveform with decimation epoch N_s.

The transfer function of the moving-average filter is:

$$A(z) = z^{N_A-1} + z^{N_A-2} + \ldots + z+1 = \frac{z^{N_A}-1}{z-1}.$$

The numerator of A(z) can be factored into

$$\prod_{i=1}^{N_A} (z-z_i).$$

The zeros, z_i, of the numerator of A(z) have unit magnitudes and arguments,

$$\arg z_i = \frac{2\pi(i-1)}{N_A},$$

for $i = 1, \ldots, N_A$. It is clear that the first factor cancels with the denominator giving

$$A(z) = \prod_{i=2}^{N_A} (z-z_i).$$

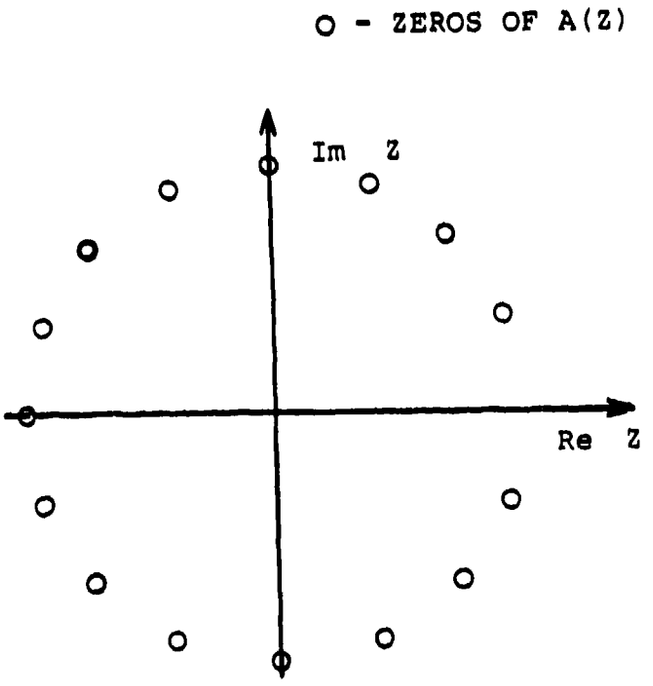

Figure E-1. Z-plane plot of zeros of preprocessing filter for N_A = 16

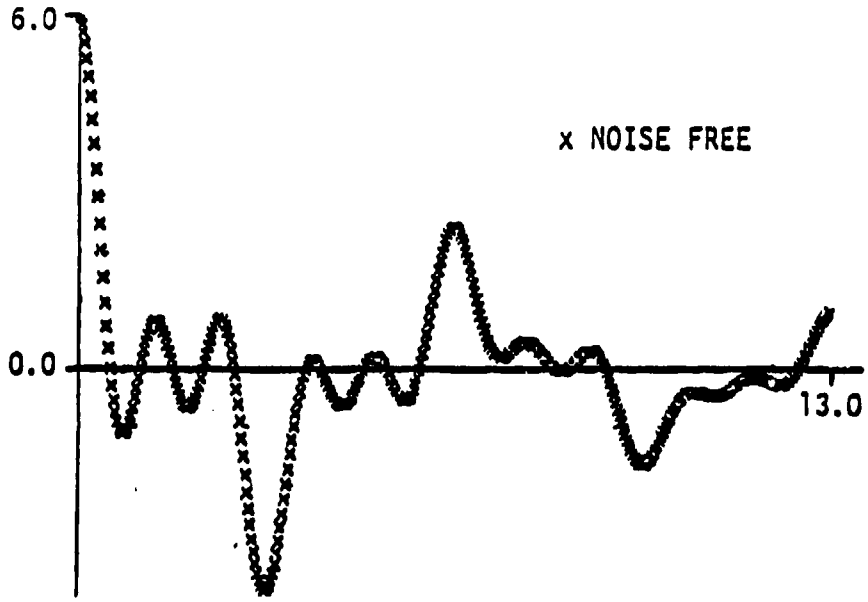

Figure E-2

a) Noise free test signal from 6 poles

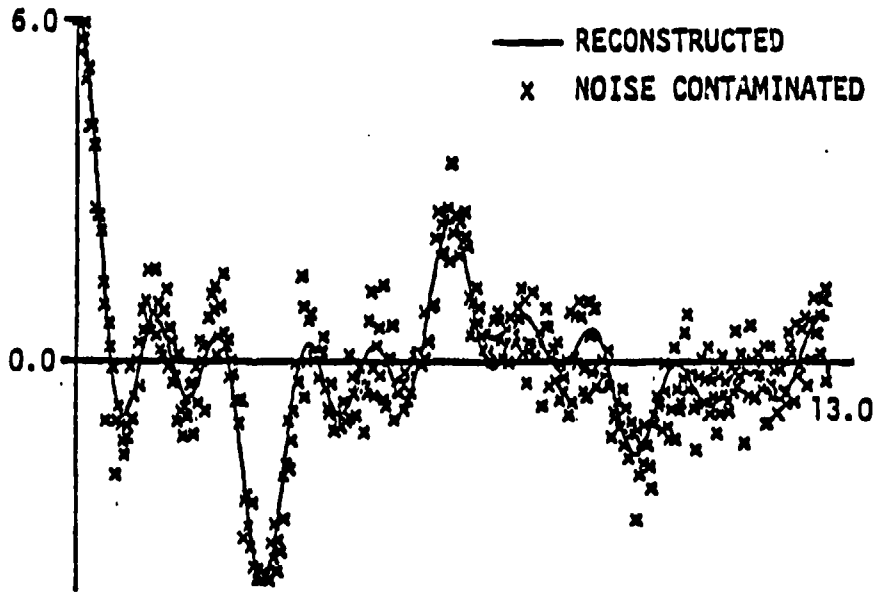

Figure E-2

b) Noise contaminated signal and reconstructed signal using the method of redundant averaging with parameters M=400, N_S=16, N_A= 160.

Table E.1 True poles and results of five Monte Carlo runs for the redundant averaging example shown in Figure E.1.

REAL PARTS OF POLES

TRUE POLES	MONTE CARLO RUNS					AVER.	σ	% DEV.
	1	2	3	4	5			
-.082	-.097	-.127	-.122	-.137	-.050	-.107	.033	30
-.147	-.198	-.157	-.239	-.162	-.177	-.187	.086	27
-1.83	-.129	-.103	-.227	-.562	-.332	-.271	.185	44

IMAGINARY PARTS OF POLES

TRUE POLES	MONTE CARLO RUNS					AVER.	σ	% DEV.
	1	2	3	4	5			
.926	.957	.904	.884	.942	.936	.925	.030	-.1
2.874	2.950	2.913	2.874	2.889	2.882	2.902	.031	1
4.835	4.721	4.899	5.006	5.065	4.803	4.899	.141	1.3

A(z) then has N_A zeros which are evenly spaced around the unit circle but the zero at z=1 is canceled. Since there is never a zero at z=1, the moving-average filter can be viewed as a type of low-pass filter. A z-plane plot of the zeros is shown in Figure E.1.

As an example of the application of this method consider the noise-free waveform shown in Figure E.2a. This waveform consists of 400 data points representing a sixth order exponential function generated with the poles listed in Table E.1. This data was then corrupted by adding white noise with a standard deviation, σ, of 0.5. Figure E.2b shows the noise contaminated signal. This signal has a signal to noise ratio of 15.6 dB where the signal to noise ratio is defined as

$$S/N = 20 \log \frac{R_{peak}}{2\sigma}$$

where R_{peak} is the peak amplitude of the transient signal. Five Monte Carlo trials were run on this data using the redundant averaging method. The model order was selected to be 8 (two more than the known order) and N_S and N_A were chosen to be 16 and 160 respectively. Figure 3.2b shows the reconstructed waveform obtained from one of the Monte Carlo trials. Note how good the fit is considering the signal to noise ratio was 15.6 dB. Table E.1 lists the results of the five trials and shows the pole averages, standard deviations and per cent deviations. It should also be noted that the signal to noise ratio compared to each pole residue is only 6 dB. Hence from poles with a 6 dB information content we were able to recover them very accurately.

The redundant averaging scheme has two difficulties or limitations that can cause the method to be less effective:

1. If the sampling rate in the preprocessed waveform is sufficiently low, the higher frequency poles can be folded, perhaps several times, about the Nyquist frequency.

2. The method can null modes in the preprocessed waveform that exists in the raw waveform.

The frequency folding phenomenon introduces an ambiguity in that it is not known how many times a particular pole has been folded about the Nyquist frequency. Hence, N_s possible poles are introduced for each extracted pole by the redundant averaging scheme where

$$N_s = \frac{(\Delta T)\ \text{preprocessed}}{(\Delta T)\ \text{raw}}$$

Of course, if the highest frequency mode of the waveform is known to be lower in frequency than the preprocessed waveform's Nyquist frequency, the ambiguity is resolved but, in general, this will not be the case.

The mode nulling can occur if the averaging parameters are such that the zeros of the resulting low pass filter cancel poles of the signal itself. That is, if we define the preprocessed waveform P(z) as

$$P(z) = A(z)\ \frac{N(z)}{D(z)}$$

resulting from the averaging process A(z) operating on the signal $\frac{N(z)}{D(z)}$, then if A(z) and D(z) each have terms in common they can tend to cancel each other.

APPENDIX F

COLUMN PRONY'S METHOD

In this method the z-plane estimates of the poles are the roots of the polynomial

$$\alpha_0 + \alpha_1(z^N) + \alpha_2(z^N)^2 + \ldots + \alpha_N(z^N)^N = 0$$

The α_i are determined from the system,

$$\begin{bmatrix} y_0 & y_N & \cdots & y_{N^2-N} \\ y_1 & y_{N+1} & \cdots & y_{N^2-N+1} \\ \vdots & \vdots & & \vdots \\ y_{N-1} & y_{2N-1} & \cdots & y_{N^2-1} \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{N-1} \end{bmatrix} = - \begin{bmatrix} y_{N^2} \\ y_{N^2+1} \\ \vdots \\ y_{N^2+N-1} \end{bmatrix}$$

where α_N is assumed to be one and $\{y_0, y_1, \ldots, y_{N^2+N-1}\}$ denotes the sequence for the waveform containing N^2+N samples. It should also be noted that the polynomial has N^2 roots only N of which are related to estimates of the true constituent poles of the waveform.

Column Prony's method offers a means of combining N^2+N data points compared to the 2N data points of the standard Prony's method. The column Prony's method can either be used in the least-squares version or the curve-fitting version. If the curve-fitting version of column Prony's method is used, the resulting parameter estimates are unbiased. Unfortunately the method yields N^2 pole estimates only N of which are the true poles. It appears then that column Prony's method leads to the same problem that increasing the model order led to in the standard Prony's method: an ambiguity in the identity of the true poles. Moreover, this method has an additional problem: the matrix can become nearly singular

even if the model order is lower than or equal to true order. This phenomenon is similar to the singularity of the standard Prony's method when the model order equals the waveform order and the highest frequency is equal or nearly equal to the Nyquist frequency.

APPENDIX G

EVAN'S AND FISCHL'S METHOD

In Evan's and Fischl's method [G.1] they define a "true error" sequence $\{e_k, k=0, \ldots, M-1\}$ as the error between the given waveform and the "fitted" waveform. They then proceed to define the "equation error" sequence $\{d_k, k=0, \ldots, M-1\}$. They further proceed to define a relation between the equation error and the true error:

$$\underline{e} = W\underline{d}$$

where

$$\underline{e} \equiv [e_0 \; e_1 \; \cdots \; e_{M-1}]^T \quad ,$$

$$\underline{d} \equiv [d_0 \; d_1 \; \cdots \; d_{M-N}]^T \quad ,$$

and $W = A[A^TA]^{-1}$

where

$$A = \begin{bmatrix} \alpha_0 & 0 & \cdots & 0 \\ \alpha_1 & \alpha_0 & \cdots & \vdots \\ \vdots & \vdots & & 0 \\ \alpha_N & \alpha_{N-1} & \cdots & \alpha_0 \\ 0 & \alpha_N & & \vdots \\ \vdots & \vdots & & \alpha_{N-1} \\ 0 & 0 & \cdots & \alpha_N \end{bmatrix}$$

M is the number of samples, N is the number of poles, $\alpha_N=1$, and the α_i are the coefficients of Prony's difference equation defined in Section 2, Volume I.

By using this relation between the two errors they describe two iterative procedures aimed at minimizing the "true" error criterion:

$$\sum_{k=1}^{M-1} e_k^2 \quad .$$

The first procedure uses

$$\underline{a}^i = [VG]^{-1} V \underline{g}$$

where

$$V \equiv G^T W(\underline{a}^{i-1})^T W(\underline{a}^{i-1}) \quad ,$$

$$\underline{a}^i \equiv [\alpha_{N-1}^i \ \alpha_{N-2}^i \ \cdots \ \alpha_0^i]^T \quad ,$$

$$\underline{g} \equiv [-y_N \ -y_{N+1} \cdots \ -y_{M-1}]^T \quad ,$$

$$G = \begin{bmatrix} y_{N-1} & \cdots & y_0 \\ y_N & \cdots & y_1 \\ \vdots & & \vdots \\ y_{M-2} & \cdots & -y_{M-N-1} \end{bmatrix} ,$$

i is the iteration number, and y_i denotes the i^{th} sample of the response
The second procedure uses:

$$\underline{a}^i = [UG]^{-1} U \underline{g}$$

where

$$U = [L(\underline{a}^{i-1})^T + G^T W(\underline{a}^{i-1})^T] W(\underline{a}^{i-1}) \quad ,$$

$$L(\underline{a}) = [\ell_1(\underline{a}) \ \ell_2(\underline{a}) \ \cdots \ \ell_N(\underline{a})] \quad ,$$

and

$$\ell_k(\underline{a}) = \left[\frac{\partial}{\partial \alpha_{N-k}} W(\underline{a})\right] \underline{d} \quad .$$

It can be shown that

$$\frac{\partial}{\partial \alpha_{N-k}} \quad W(\underline{a}) =$$

$$\left\{ \left[\frac{\partial}{\partial \alpha_{N-k}} \quad A(\underline{a}) \right] - W(\underline{a}) \left(\left[\frac{\partial}{\partial \alpha_{N-k}} \quad A(\underline{a}) \right]^T A(\underline{a}) \right.\right.$$

$$\left.\left. + A(\underline{a})^T \left[\frac{\partial}{\partial \alpha_{N-k}} \quad A(\underline{a}) \right] \right) \right\} \left[A(\underline{a})^T A(\underline{a}) \right]^{-1}$$

and $[(\partial/\partial \alpha_{N-k}) \; A(\underline{a})]$ is simply the matrix $A(\underline{a})$ with ones replacing the α_{N-k} with all other elements zero.

The following observations can be made about the method:

1. It produces optimal pole estimates in the sense that it minimizes "true error". This means that the mean-square error between the given waveform and the fitted waveform is minimized. These optimal pole estimates are obtained only in the second iteration phase.

2. The matrix A^TA must be inverted on each iteration of both the first and second procedures. When the waveform has over 100 samples, A^TA becomes very large, and hence, expensive to invert. Consequently, the method is prohibitively expensive when the waveform has over 100 samples. This method has yet to be evaluated by tests on noisy data. Nothing is presently known about its convergence characteristics of its tolerance of noise.

The following example illustrates the optimal estimates.

Example

$N = 1,\ M = 3$

$\underline{y} = [2\ 1\ 2]$ (waveform to be approximated)

$G = \begin{bmatrix} 2 \\ 1 \end{bmatrix}\ \underline{g} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$

Initial parameters: $\alpha_0 = 1$, $\alpha_1 = -1$, $\beta_0 = \frac{5}{3}$. These are the optimal parameters, therefore the method should converge immediately.

$$A = \begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$\left[A^T A\right]^{-1} = \left\{ \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} \right\}^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{bmatrix}$$

$$\underline{d} = G\underline{a} - \underline{g} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \begin{bmatrix} -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$W = \begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{bmatrix} = \begin{bmatrix} -2/3 & -1/3 \\ 1/3 & -1/3 \\ 1/3 & 2/3 \end{bmatrix}$$

$$\frac{\partial}{\partial\, \alpha_0} A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\frac{\partial}{\partial \alpha_0} W = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} -2/3 & -1/3 \\ 1/3 & -1/3 \\ 1/3 & 2/3 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} \right. \right.$$

$$\left. \left. + \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \right) \right\} \begin{bmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{bmatrix}$$

$$\frac{\partial}{\partial \alpha_0} W = \begin{bmatrix} 0 & 0 \\ 2/3 & 1/3 \\ 1/3 & 2/3 \end{bmatrix}$$

$$V = [5/3 \quad 5/3]$$

$$\hat{\alpha}_0 = \left\{ [5/3 \quad 5/3] \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}^{-1} [5/3 \quad 5/3] \begin{bmatrix} -1 \\ -2 \end{bmatrix}$$

$$= \left(\frac{1}{5} \right) (-5) = -1 \quad .$$

Therefore the updated value of α_0 is equivalent to its old value, hence the method has converged at the optimal parameters for the second iteration phase.

Reference

G.1 A.G. Evans and R. Fischl, "Optimal Least-Squares Time-Domain Synthesis of Recursive Digital Filters", IEEE Trans. Audio and Electro-acoustics, AU-21, February 1973, pp. 61-65.

APPENDIX H

CONSTRAINED PRONY's METHOD

H.1 Uses of a Constrained Prony Method

Many times, in the application of the Prony algorithm some of the poles are known a priori. It would be very useful if the Prony algorithm could be constrained in such a manner that the knowledge of the known poles is used in extracting the unknown poles. The known poles could be poles of the driving function which are known from knowledge of the Laplace transform of the driving function, system poles which are known from previous Prony analysis or other techniques, or poles introduced to model the noise.

It is well known that for certain data sets Prony's algorithm has some difficulty in extracting the true poles that are contained in the data. This problem is generally related to the noise in the data but will not be discussed here. Any method by which the accuracy of the true poles can be increased will be very useful.

Logic would tell one that making use of known information should increase the accuracy of a calculation. Hence, if use is made of known poles in the data it would seem reasonable to assume that some of the instability in Prony's method would be alleviated. The proof of the validity of this statement rests on the actual implementation of the constrained Prony algorithm and the results compared for the same data analyzed by the unconstrained method.

In addition to aiding in increasing the accuracy of the true poles it might be possible to introduce random poles which will model the noise. In that manner, the poles that are known a priori are not the poles which we seek. The difficulty with this, if it should work, is that we need to know the rank of the system so that we can introduce the proper number of noise poles.

Another asset of having a contrained Prony's method is that it can be used to aid in deconvolution. The following discussion follows directly from work performed in Reference [4.2].

Assume that a system is excited by a driving function which can be represented in the form

$$G(t) = \sum_{j=1}^{M} g_j e^{s_j t} u(t). \qquad (H.1)$$

It is presumed here that the driving function poles s_j and the associated residues g_j are known. These can be determined analytically if the analytical form of the driver is known or can be determined from a Prony's method fit to measurement data of the driving function.

Now assume that the response of the system to the driving function G(t) is

$$R(t) = \sum_{i=1}^{M+N} r_i e^{s_i t} u(t) \qquad (H.2)$$

and that the impulse response of the system is

$$H(t) = \sum_{k=1}^{N} h_k e^{s_k t} \quad . \qquad (H.3)$$

Hence, there are N system poles and residues and M poles in the driving function. Of course, the response function R(t) can be written as the convolution of the driving function G(t) with the impulse response H(t). That is,

$$R(t) = H(t)*G(t) \quad , \qquad (H.4)$$

where * denotes convolution.

What is usually desired is to obtain the N system poles and residues of (H.3). This must be done by deconvolution of the driving function (H.1) from (H.2). If the r_i and s_i are known, the M+N values of s_i contain the M values s_j. The g_j and the s_j are known, then the N values of h_k and s_k can be determined analytically.

It can be shown that the response function can be written in terms of the driving function and the impulse response as

$$R(t) = \sum_{j=1}^{M} \left(g_j \sum_{k=1}^{N} \frac{h_k}{s_j - s_k}\right) e^{s_j t} - \sum_{k=1}^{N} \left(h_k \sum_{j=1}^{M} \frac{g_j}{s_j - s_k}\right) e^{s_k t} \quad . \qquad (H.5)$$

Hence, the response function residues can be defined as

$$r_i = g_j \sum_{k=1}^{N} \frac{h_k}{s_j - s_k} \quad , \quad \text{for } i = 1 \ , \ M \qquad (H.6a)$$

$$r_i = -h_k \sum_{j=1}^{M} \frac{g_j}{s_j - s_k} \quad , \quad \text{for } i = M+1 \ , \ N+M \quad . \qquad (H.6b)$$

From (H.6b) the residues of the impulse response are written as

$$h_k = \frac{r_i}{\sum_{j=1}^{M} \frac{g_j}{s_j - s_k}} \quad , \qquad \begin{matrix} k = 1 \ , \ N \\ i = M+k \end{matrix} \quad . \qquad (H.7)$$

Thus the residues or amplitudes of the impulse response can be obtained from the known values r_i, g_j, s_j, s_k. Of course, since the M values of s_j are known and the M+N values of s_i are presumed to contain the M values of s_j, then the N values of s_k can be determined by inspection.

Prony's method, however, will not necessarily give the true values of the driving function poles in the extracted poles of the response function. Hence a constrained Prony's method is required so that analytical deconvolution of (H.7) can be obtained.

Work performed so far implements Method 1 which is described below. It was found that this method works perfectly as long as no noise is present in the signal. As soon as noise is introduced in the signal and a least-squares method is used the matrix (H.17a) is corrupted with noise which through the least-squares process also corrupts the constrained parameter. Experiments have confirmed this observation. This suggests that the constraining Method 1 may work well in noisy data if the curve-fitting Prony's method is used. This has not been tested as yet.

Method 2, forcing the polynomial root solver to find certain roots, has also not been tested. This is because if the coefficients are all corrupted by noise through the least-squares procedure then subtracting any given poles out of the polynomial will force the remaining poles to carry the burden of all the noise.

H.2 Method 1

In the implementation of Prony's method, an Nth order polynomial is solved for its roots. The order of the polynomial, N, is the number of poles being sought in the transient data. If the coefficients of the polynomial are denoted as α then the polynomial can be expressed as per Reference H.1 as

$$\alpha_0 + \alpha_1 Z^1 + \alpha_2 Z^2 + \ldots + \alpha_N Z^N = 0 \qquad \text{(H.8)}$$

where α_N is usually set equal to unity. The N roots, Z_i, are defined as

$$Z_i = e^{\Delta t s_i} \tag{H.9}$$

with s_i being the poles sought, and Δt being the time step size used in the analysis.

If the value of one or more of the poles is known - that is, we know some of the Z_i - then the Z_i can be substituted into (H.8). For example, if s_i or Z_i is known, then (H.8) can be written as

$$\alpha_0 + \alpha_1 Z_i + \alpha_2 Z_i^2 + \ldots + \alpha_N Z_i^N = 0 \quad . \tag{H.10}$$

The N+1 polynomial coefficients α are solved for in Prony's method by solution of the difference equation

$$\sum_{p=0}^{N-1} \alpha_p I_{p+K} = -I_{N+K}, \; k=0,1,\ldots,\gamma-1 \quad , \quad \gamma=M-N \quad . \tag{H.11}$$

The I_{p+K} and I_{N+K} are the samples of the transient signal being analyzed and M is the total number of samples being used. The value of M must be at least equal to 2N to give N sets of equations in the N unknowns α_p. However, if the value of a pole is known, then one of the N equations can be equation H.10 and N-1 equations of the form of H.11 can be written.

If L poles are known a priori, the L equations of the form of (H.10) can be written as

$$\sum_{p=0}^{N-1} \alpha_p Z_\ell^p = - Z_\ell^N, \; \ell = 1, 2, \ldots, L \tag{H.12}$$

and $\gamma=M-N-L$ equations can be written in the form of (H.11)

$$\sum_{p=0}^{N-1} \alpha_p I_{p+K} = -I_{K+N} \quad , \quad k=0,1,\ldots,\gamma-1 \quad , \quad \gamma=M-N-L \quad . \tag{H.13}$$

Hence there are still $\gamma=M-N$ total equations to solve for the N values of α_p however the system is constrained by the knowledge of the location of L poles. As is usually done in Prony's method, if M=2N then the set of equations is inverted and solved. If M>2N then a pseudo-inverse procedure is used.

Using the matrix notation of Reference H.1, if M=2N and L poles are known, then we solve the equation

$$AB = C \tag{H.14}$$

where A is a square matrix defined as

$$A = \begin{bmatrix} 1 & z_1 & z_1^2 & \cdots & z_1^{N-1} \\ 1 & z_2 & z_2^2 & \cdots & z_2^{N-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & z_L & z_L^2 & & z_L^{N-1} \\ I_0 & I_1 & I_2 & \cdots & I_{N-1} \\ I_1 & I_2 & I_3 & \cdots & I_N \\ \vdots & \vdots & \vdots & & \vdots \\ I_{N-1-L} & I_{N-L} & I_{N-L+1} & \cdots & I_{2N-L-2} \end{bmatrix} \tag{H.15a}$$

and B and C are vectors defined as

$$B = \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{N-1} \end{bmatrix} \qquad (H.15b)$$

$$C = - \begin{bmatrix} Z_1^N \\ Z_2^N \\ \vdots \\ Z_L^N \\ I_N \\ I_{N+1} \\ \vdots \\ I_{2N-1-L} \end{bmatrix} \qquad (H.15b)$$

If M>2N and L poles are known, then the solution takes on a pseudo-inverse or least-squares form as

$$A^T AB = A^T C \qquad (H.16)$$

Where A^T is the transpose of the matrix A and A is now a rectangular matrix of the form

$$A = \begin{bmatrix} 1 & Z_1 & Z_1^2 & \cdots & Z_1^{N-1} \\ 1 & Z_2 & Z_2^2 & \cdots & Z_2^{N-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & Z_L & Z_L^2 & \cdots & Z_L^{N-1} \\ I_0 & I_1 & I_2 & \cdots & I_{N-1} \\ I_1 & I_2 & I_3 & \cdots & I_N \\ \vdots & \vdots & & & \\ I_{M-N-1-L} & I_{M-N-L} & & \cdots & I_{M-L-2} \end{bmatrix} \qquad (H.17a)$$

$$B = \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{N-1} \end{bmatrix} \qquad (H.17b)$$

$$C = - \begin{bmatrix} z_1^N \\ z_2^N \\ \vdots \\ z_L^N \\ I_N \\ \vdots \\ I_{M-1-L} \end{bmatrix} \qquad (H.17c)$$

Consider the simple example of a two pole system

$$f(t) = 2e^{s_1 t} + 3e^{s_2 t}$$

where $s_1 = 0$ and $s_2 = -4$. Assume that s_1 is known, thus giving $Z_1 = 1.0$. Also, let f(t) be sampled at $\Delta t = 0.2$ seconds, giving a data set as

t	f(t)
0	$I_0 = 5$
0.2	$I_1 = 3.3480$
0.4	$I_2 = 2.6057$
0.6	$I_3 = 2.2722$
0.8	$I_4 = 2.1223$

Using the constrained Prony's method in the square system form of Equation (H.14) and (H.15) yields

$$\begin{bmatrix} 1 & 1 \\ I_0 & I_1 \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix} = - \begin{bmatrix} 1 \\ I_2 \end{bmatrix}$$

or

$$\begin{bmatrix} 1 & 1 \\ 5 & 3.3480 \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix} = - \begin{bmatrix} 1 \\ 2.6057 \end{bmatrix}$$

The solution of this set of equations gives $\alpha_1 = 1.4493$ and $\alpha_0 = 0.4493$ so that the polynomial can be written as

$$Z^2 - 1.4493\ Z + 0.4493 = 0 \quad .$$

The roots of this polynomial are $Z_1 = 1.0$ and $Z_2 = 0.4493$, giving poles of $s_1 = 0.0$ and $s_2 = 4.0003$. The error in s_2 is due to truncation error. Note that the constrained pole was returned exactly.

H.3 Method 2

Another approach which can be used to constrain Prony's method to use information about known poles is the modification of the polynomial root finding routine MULLER. Once the unconstrained Prony's method calculates the coefficients of the polynomial (H.8) and if some of the roots of the polynomial are known a priori, then the locations of those roots can be passed to MULLER and it will not have to search for those roots. Since the root finding routine is very time consuming, the knowledge of the location of any of the roots will presumably save computation time.

A possible flaw with this approach is that MULLER is forced to presume roots of the polynomial when those exact roots may not be contained in the polynomial. That is, if the polynomial has not been constrained to contain the known roots, as per Section H.2, then the known roots will

not necessarily be contained in it. Forcing an unconstrained polynomial to have certain roots could grossly perturb the location of the other roots being sought.

H.4 Method 3

The obvious solution to the flaw presented in Section H.3 is to use both the methods of H.2 and H.3 simultaneously. That is, the polynomial is constrained to contain the known poles as outlined in Section H.2. Then the polynomial root finding routine MULLER can be modified to extract the known roots from the polynomial before it begins its search for the unknown roots.

It is felt that this approach will give the best accuracy and will speed up the calculations since the order of the polynomial is effectively reduced.

References

H.1 Van Blaricum, M.L., Techniques for Extracting the Complex Resonances of a System Directly from its Transient Response, Ph.D. Dissertation, December 1975, University of Illinois, Department of E.E.

H.2 A. J. Poggio, M.L. Van Blaricum, E.K. Miller and R. Mittra, "Evaluation of a Processing Technique for Transient Data," IEEE Trans. on Ant. and Prop., pp. 165-173, January 1978.

APPENDIX I

REVERSING THE WAVEFORM IN TIME TO ELIMINATE EXTRANEOUS RESONANCES

Reversing the waveform in time depends on the statistical properties of the noise which do not change when the waveform is reversed. The poles that result from the noise, therefore, are not altered by reversal in time while the true poles are negated or flipped through the origin.

If the noise level is high the noise poles only approximately remain the same under time reversal and the true poles only approximately reflect through the origin. If time reversal is attempted for curve-fitting Prony's method it is found that all poles flip precisely. Therefore, this method is not effective for curve-fitting Prony's method.

(Curve-fitting Prony's method uses the solution to the inhomogeneous system $\bar{\bar{Q}}\bar{X}=q$, in the notation of Volume I, where $\bar{\bar{Q}}$ is a square matrix.)

The waveform of Figure I.1 was used in a numerical example of the time-reversal method. Least-squares Prony's method was applied to the waveform. Estimates at the S-Plane poles were found using the inhomogeneous solution which is defined in Volume I, Section 2 as

$$\bar{x}_I = [\bar{\bar{Q}}^T\bar{\bar{Q}}]^{-1}\,\bar{\bar{Q}}^T\bar{q}.$$

The waveform consists of 100 samples and was corrupted with uncorrelated, Gaussian-distributed noise with a standard deviation of 0.1. Figure I.2 displays the poles obtained from the forward and reversed waveforms. The dimensions of the (NXn)-dimensional matrix $\bar{\bar{Q}}$ in this example are M=76 and n=24. The estimates of the true poles are quite accurate. Figure I.3 displays the poles obtained for the same waveform but different matrix dimensions. The matrix dimensions are M=60 and n=40. In this case, the extraneous poles do not all remain in the left-half plane which is attributed to using an overly square matrix.

If the matrix is too "thin", of M/n >>1, the estimates of the true poles become inaccurate. If the matrix is too "fat", or M/n ≈ 1, the extraneous poles do not all remain in the left-half plane. The matrix shape where M/n ≈ 3 appears to be the best shape for good estimates and keeping the extraneous poles in the left-half plane.

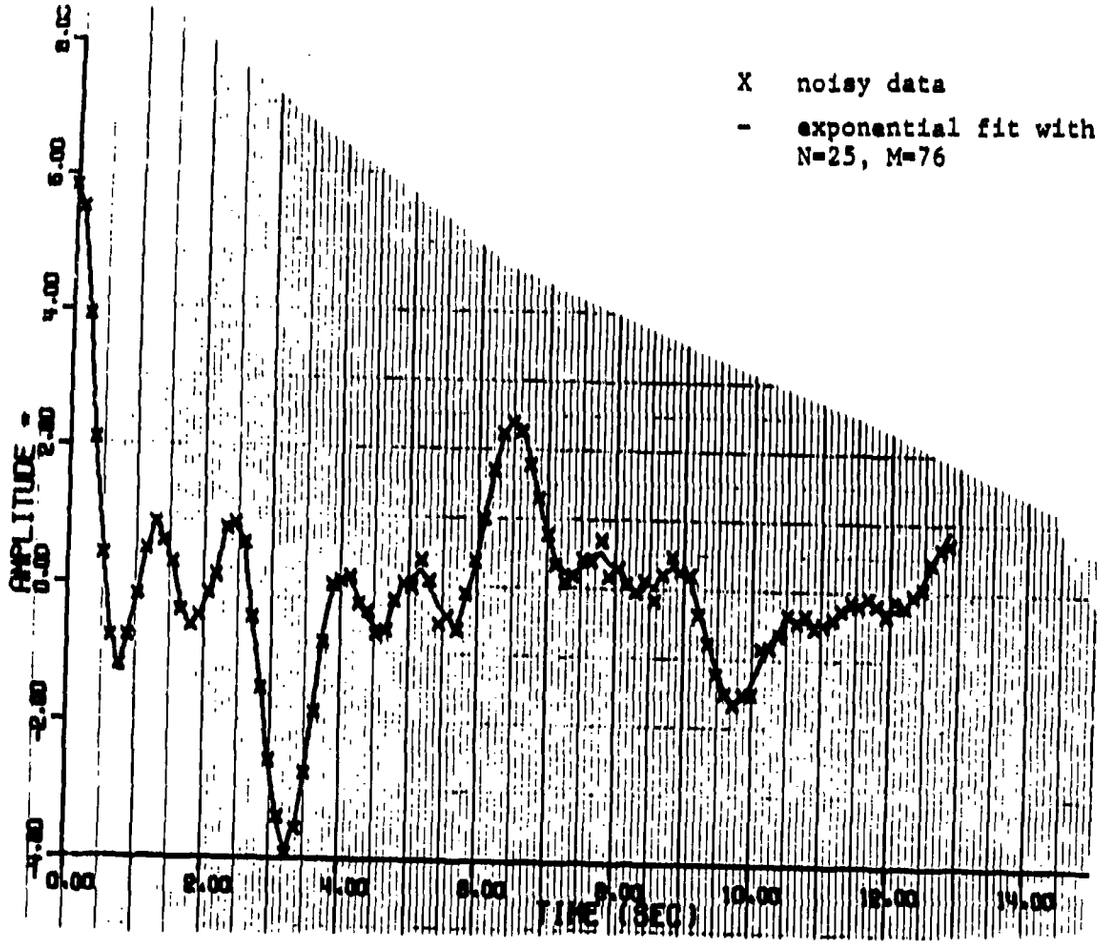

Figure I-1. Waveform used in the numerical example.

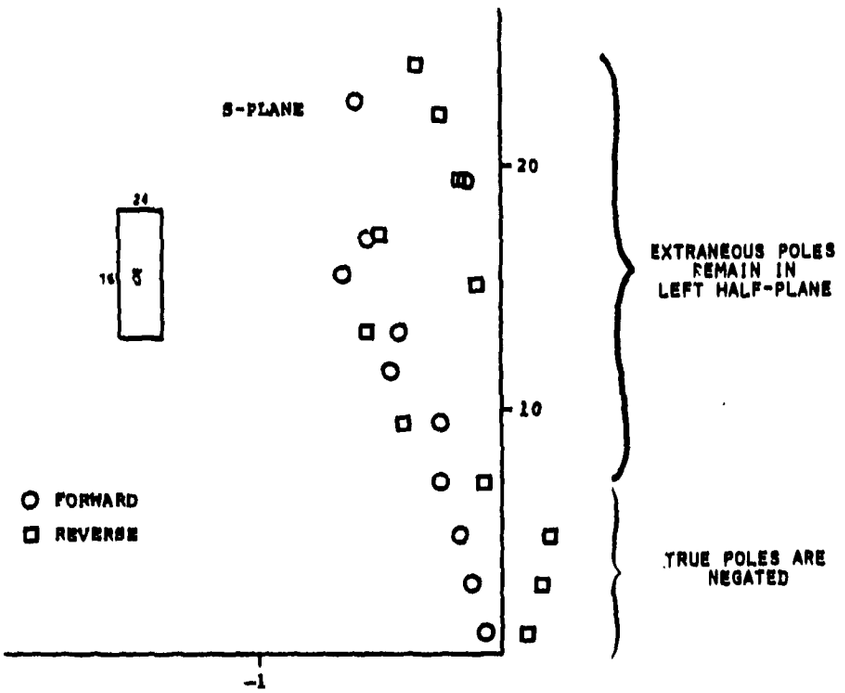

Figure I-2 Effect of reversing the waveform.

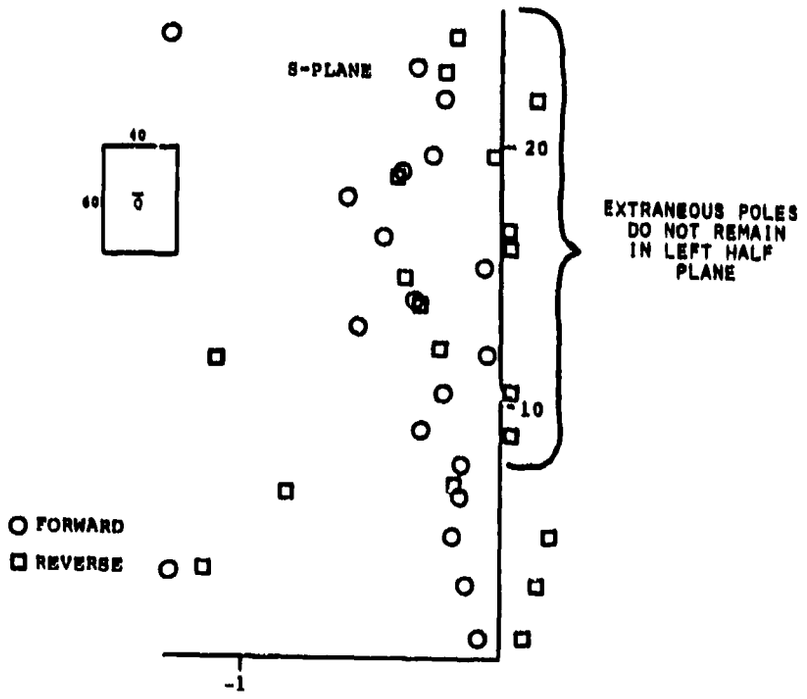

Figure I-3 Effect of an overly-square data matrix.

Therefore, it appears that if we choose M/n ≥ 3, then time reversal can be used to distinguish between the true poles and the extraneous poles in relatively high noise levels.

The above two examples show that there is a trend for the noise poles in the least-squares Prony's method to occupy a higher frequency portion of the S-plane and to be highly damped. This trend was studied by letting the least-squares Prony's method operate on a waveform consisting of only Gaussian-distributed uncorrelated noise - no signal. The poles resulting from this example are highly damped and evenly distributed in the z-plane approximately around a circle within the unit circle.

This behavior can be explained by the fact that the polynomial coefficients, except α_N which is set to one, all tend to zero for the least-squares method operating on uncorrelated noise. The polynomial then tends toward $z^N = \varepsilon$ where ε tends to zero. The roots of this polynomial, z_i, have magnitudes

$$|z_i| = |\varepsilon|^{1/N}$$

and arguments

$$\arg z_i = \frac{\arg \varepsilon + 2\pi(i-1)}{N}$$

for $i = 1, \ldots, N$.

It then follows that the poles should be highly damped, since $|z_i| \to 0$ if $|\varepsilon| \to 0$, and that the poles are evenly distributed about the z-plane. However, for curve-fitting Prony's method the α_i, $i = 0, \ldots, N-1$ do not tend to zero and indeed this phenomenon is not observed in tests using curve-fitting Prony's method on all noise.

This behavior of the noise poles in least-squares Prony seems to remain approximately the same even if the waveform is not entirely noise. The trend that is seen is that the noise poles occupy the higher frequencies, are highly damped and evenly distributed between the higher frequencies; while the true poles are approximately at their uncorrupted locations and occupy the lower frequencies. That is, it appears as though the noise poles are "crowded" away from the lower frequencies or, perhaps more accurately, the lower frequency noise poles become lower frequency signal poles when the waveform is no longer entirely noise.